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The external multiplication for the Conley index ☆

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Abstract

We construct an additional operation of the external multiplication on the cohomological Conley index defined by Mrozek for discrete semidynamical systems. The construction is based on the notion of the Conley index over a phase space introduced by Szybowski. We show how to apply the external multiplication to solve the problem of continuation of two isolated invariant sets and illustrate it by an example.

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1. Introduction

The Conley index theory has been developing in a rapid way. The main reason for this is that the Conley index is a very useful homotopy invariant for an isolated invariant set. Its advantage is that it allows to tell more about the dynamics of an isolated invariant set than any other indices. However, the more we want to know about a dynamical system, the more complicated index we apply. Unfortunately, it may cause problems with calculating the index in a concrete situation. Sometimes simpler indices seem to be more effective, but what if they do not answer the question of our interest?

The problem described above appears sometimes if we want to apply the Conley index over a base defined in [7] for flows. The index generalizes the classical one from [1] because it distinguishes the way how an isolated invariant set is situated in a phase space. Unfortunately, its definition is quite complicated, so in [9] an interesting improvement was presented. The main idea was to use the more general index from [7] to define the so-called external multiplication as an additional operation for the cohomology version of the classical index.

Recently an analogous generalization of Conley-type indices for discrete semidynamical systems, the so-called Conley index over a phase space, has been presented in [10] and [11]. As the discrete case is more difficult to consider, the number of necessary computations even increased. A natural attempt was to use the idea of [9] to improve the Conley index over a phase space.

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Using the index over a phase space we endow the cohomology Conley index defined in [6] with the structure of the left module (Theorem 5.6). This structure is invariant under continuation, so it allows to pass the problem of continuation for isolated invariant sets from topology to algebra with hardly any loss of information. We illustrate it by a nontrivial example.

The organization of the paper is as follows. In Section 2 we recall the basic notions from the theory of isolated invariant sets. Section 3 contains the definition of the so-called M-equivalence. This relation was crucial to give the definition of the Conley index over a phase space in [11], which is recalled in Section 4. The last section introduces the operation of the external multiplication and gives an example of its application in the study of dynamical systems.

2. Isolated invariant sets

2.1. Continuous case

Let X be a locally compact metric space, $\rho : X \times \mathbb{R} \to X$—a continuous dynamical system. To simplify notation, for $x \in X$ and $a, b, t \in \mathbb{R}$ we will write $x \cdot t$ instead of $\rho(x, t)$ and $x \cdot [a, b]$ instead of $\rho(x, [a, b])$.

For an arbitrary set $N \subseteq X$ we define the set

$$\operatorname{Inv} N = \{x \in N\colon\ x \cdot \mathbb{R} \subseteq N\},$$

which will be called an invariant part of N. A set $N \subseteq X$ is called an invariant set when $N = \operatorname{Inv} N$. A compact set $N \subseteq X$ is called an isolating neighborhood for $S := \operatorname{Inv} N$ if $S \subseteq \operatorname{int}(N)$. The set S is called an isolated invariant set.

Recall the notion of an isolating block from [8].

Let $\Sigma \subset X$. If for a given $\delta > 0$ the map

$$\rho : \Sigma \times (-\delta, \delta) \ni (x, t) \mapsto x \cdot t \in X$$

is a homeomorphism onto image, then Σ is called a local section.

Definition 2.1. Let B be the closure of an open set in X and let Σ^+, Σ^- be two disjoint local sections satisfying:

(a) $[(\mathrm{cl}, \Sigma^\pm) \setminus \Sigma^\pm] \cap B = \emptyset$.
(b) $\Sigma^+ \cdot (-\delta, \delta) \cap B = (\Sigma^+ \cap B) \cdot [0, \delta)$.
(c) $\Sigma^- \cdot (-\delta, \delta) \cap B = (\Sigma^- \cap B) \cdot (-\delta, 0]$.
(d) If $x \in (\operatorname{bd} B) \setminus (\Sigma^- \cup \Sigma^+)$, then there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $x \cdot [-\varepsilon_1, \varepsilon_2] \subset \operatorname{bd} B$ and $x \cdot -\varepsilon_1 \in \Sigma^+$, $x \cdot \varepsilon_2 \in \Sigma^-$.

A set B satisfying the above conditions is called an isolating block for a dynamical system ρ. A number δ is called a collar size for B.

For an isolating block B we define the exit set $B^- := B \cap \Sigma^-$.

Proposition 2.2. *An isolating block B is an isolating neighborhood for $S := \operatorname{Inv} B$.*

2.2. Discrete case

Let X be a locally compact metric space, $f : X \to X$—a continuous map which generates a semidynamical system.

For an arbitrary set $N \subseteq X$ we define the set

$$\operatorname{Inv} N = \bigl\{x \in N\colon\ \exists\{x_k\}_{k\in\mathbb{Z}} \subseteq N\ x_0 = x \text{ and } f(x_k) = x_{k+1} \text{ for } k \in \mathbb{Z}\bigr\},$$

which will be called an invariant part of N. A set $N \subseteq X$ is called an invariant set when $N = \operatorname{Inv} N$. A compact set $N \subseteq X$ is called an isolating neighborhood for $S := \operatorname{Inv} N$ if $S \subseteq \operatorname{int}(N)$. The set S is called an isolated invariant set.

Fix an isolated invariant set S and its isolating neighborhood N.

Definition 2.3. A pair $P = (P_1, P_2)$ of compact subsets of N, is called an index pair for S in N iff

(a) $S = \operatorname{Inv} N \subseteq \operatorname{int}(P_1 \setminus P_2)$,
(b) $f(P_1) \cap N \subseteq P_1$ and $f(P_2) \cap N \subseteq P_2$,
(c) $f(P_1 \setminus P_2) \subseteq N$.

If $N = P_1$, P will be called simply an index pair for S.

Assume $f, g : X \to X$ are continuous maps, S and T are isolated invariant sets for f and g, respectively. We say that S and T are related by continuation ($(f, S) \simeq (g, T)$), if there exists a continuous map $h : X \times [0, 1] \to X \times [0, 1]$, an isolated invariant set U for h, and maps $h_\lambda : X \to X$, $\lambda \in [0, 1]$, such that $h_0 = f$, $h_1 = g$, $S = \{x \in X\colon (x, 0) \in U\}$, $T = \{x \in X\colon (x, 1) \in U\}$, $h(x, \lambda) = (h_\lambda(x), \lambda) \in X \times [0, 1]$, for all $x \in X$ and $\lambda \in [0, 1]$.

Fix a continuous dynamical system $\rho : X \times \mathbb{R} \to X$, an isolated invariant set S for ρ, and an isolating block B for S. For $T > 0$ we define a time-T map $\rho_T := \rho(\cdot, T)$.

Theorem 2.4. *There exists a $T_0 > 0$ such that for each $T \in (0, T_0]$ a pair $P = (P_1, P_2) := (B \cup B^- \cdot [0, T_0], B^- \cdot [0, T_0])$ is an index pair for S in P_1.*

Proof. If δ is the collar size for B, then we take any $T_0 \in (0, \delta)$. Now fix $T \in (0, T_0]$.

Notice that

$$\operatorname{Inv} P_1 = \operatorname{Inv} B = S \subseteq \operatorname{int} B \setminus B^- = \operatorname{int} P_1 \setminus P_2 \subseteq \operatorname{int} P_1.$$

Thus, P_1 is an isolating block for S and condition (a) from Definition 2.3 is satisfied.

For condition (b) take $x \in P_2$. There exist $y \in B^-$ and $t_0 \in [0, T_0]$ such that $x = y \cdot t_0$. If $\rho_T(x) = x \cdot T = y \cdot (T + t_0) \in P_1$, then $T + t_0 \in [0, T_0]$ and $\rho_T(x) \in P_2$.

Finally, take $x \in P_1 \setminus P_2 = B \setminus B^-$. Assume $x \cdot T \notin P_1$. Then, from the definition of an isolating block, there exists a $t' \in [0, T]$ such that $x' := x \cdot t' \in B^- \subseteq P_2$ and $x \cdot [0, t'] \subseteq B \subseteq P_1$. We have $x' \cdot [0, T_0] \subseteq P_1$, so $T > t' + T_0 > T$, which is a contradiction. Condition (c) from Definition 2.3 is also satisfied and P is an index pair for S in P_1. □

3. *M*-equivalence

For a given topological space X we define the category of spaces over a base X (objects and morphisms), denoted by $\mathcal{S}B(X)$.

Definition 3.1.

$$\operatorname{Ob}\big(\mathcal{S}B(X)\big) = \big\{(U, r, s)\colon U \text{ is a topological space, } r : U \to X, s : X \to U \text{ are continuous, such that } r \circ s = \mathrm{id}_X\big\},$$

$$\operatorname{Mor}_{\mathcal{S}B(X)}\big((U, r, s), (U', r', s')\big) = \big\{(F, f)\colon F : U \to U', f : X \to X \text{ are continuous, such that } F \circ s = s' \circ f \text{ and } r' \circ F = f \circ r\big\}.$$

For two morphisms in $\mathcal{S}B(X)$ we define the relation $\simeq_*$ of homotopy:

Definition 3.2. $(F, f), (F', f') \in \operatorname{Mor}_{\mathcal{S}B(X)}((U, r, s), (U', r', s'))$,

$$(F, f) \simeq_* (F', f') \iff \exists H : U \times \mathbb{I} \to U', h : X \times \mathbb{I} \to X \text{ continuous:}$$
$$H \circ (s \times \mathrm{id}_{\mathbb{I}}) = s' \circ h,$$
$$r' \circ H = h \circ (r \times \mathrm{id}_{\mathbb{I}}),$$
$$H(\cdot, 0) = F, H(\cdot, 1) = F',$$
$$h(\cdot, 0) = f, h(\cdot, 1) = f',$$

where $\mathbb{I} = [0, 1]$. A pair (H, h) will be called a homotopy joining (F, f) with (F', f').

Fix $(U,r,s), (U',r',s') \in \mathrm{Ob}(\mathcal{S}B(X))$ and two morphisms $(F,f) \in \mathrm{Mor}_{\mathcal{S}B(X)}((U,r,s),(U,r,s))$ and $(F',f') \in \mathrm{Mor}_{\mathcal{S}B(X)}((U',r',s'),(U',r',s'))$.

Definition 3.3. Two objects $((U,r,s),(F,f))$ and $((U',r',s'),(F',f'))$ are M-equivalent over a base X, if $f \simeq f'$ and there exist $m,n \in \mathbb{N}$, and continuous maps $\Phi : U \to U'$, $\Psi : U' \to U$, $\varphi, \psi : X \to X$, such that $\varphi \simeq f^m$, $\psi \simeq f'^n$, and there exists a $k \in \mathbb{N}$ such that

$$\begin{aligned}
&\Phi \circ s = s' \circ \varphi, \qquad \Psi \circ s' = s \circ \psi,\\
&r' \circ \Phi = \varphi \circ r, \qquad r \circ \Psi = \psi \circ r',\\
&(\Phi \circ F, \varphi \circ f) \simeq_* (F' \circ \Phi, f' \circ \varphi),\\
&(\Psi \circ F', \psi \circ f') \simeq_* (F \circ \Psi, f \circ \psi),\\
&(\Psi \circ \Phi \circ F^k, \psi \circ \varphi \circ f^{m+n+k}) \simeq_* (F^{m+n+k}, f^{m+n+k}),\\
&(\Phi \circ \Psi \circ F'^k, \varphi \circ \psi \circ f'^k) \simeq_* (F'^{m+n+k}, f'^{m+n+k}).
\end{aligned}$$

The class of M-equivalence of $((U,r,s),(F,f))$ over X will be denoted by $[((U,r,s),(F,f))]_X$.

4. The Conley index over a phase space

Fix a locally compact metric space X, a continuous map $f : X \to X$, an isolated invariant set S for f, and an index pair $P = (P_1, P_2)$ for S.

We define $U(P)$ as the adjunction $P_1 \cup_{\mathrm{id}|_{P_2}} X$, i.e.

$$U(P) := X \times 0 \cup P_1 \times 1 / \sim,$$

where $\sim$ denotes the minimal equivalence relation such that $(x,0) \sim (x,1)$ for each $x \in P_2$. Let $[x,q]_P$ denotes the equivalence class of (x,q) in $U(P)$.

We also define two maps: $s_P : X \ni x \mapsto [x,0]_P \in U(P)$ and $r_P : U(P) \ni [x,q]_P \mapsto x \in X$.

An index space over X is a triple $(U(P), r_P, s_P)$. An index map $f_P : U(P) \to U(P)$ is given by:

$$f_P([x,q]_P) := \begin{cases} [f(x),1]_P, & \text{for } q = 1,\ x,\ f(x) \in P_1 \setminus P_2,\\ [f(x),0]_P, & \text{otherwise.} \end{cases}$$

Theorem 4.1. *(See [11, Theorem 4.6].) For any index pairs P, P' for S objects $((U(P), r_P, s_P), (f_P, f))$ and $((U(P'), r_{P'}, s_{P'}), (f_{P'}, f))$ are M-equivalent over a phase space X.*

Definition 4.2. The Conley index $\hat{h}_d(S,f)$ of an isolated invariant set S over a phase space is the M-equivalence class of the object $((U(P), r_P, s_P), (f_P, f))$ over X, for any index pair P for S:

$$\hat{h}_d(S,f) = \big[\big((U(P), r_P, s_P), (f_P, f)\big)\big]_X.$$

Theorem 4.1 shows that the Conley index of an isolated invariant set over a phase space does not depend on the choice of an index pair. Theorem 5.3 in [11] says that the index is also invariant under continuation:

Theorem 4.3 *(Continuation property). Assume $f, g : X \to X$ are continuous maps, S and T are isolated invariant sets for f and g, respectively. Then*

$$(f,S) \simeq (g,T) \Rightarrow \hat{h}_d(S,f) = \hat{h}_d(T,g).$$

5. External multiplication

In [3] and [9] the authors define an additional structure of the external multiplication, which allows to use the Conley index over a base to examine the problem of continuation. In this chapter we define an analogous operation and show how to use it in the study of dynamical systems.

The author of [6] defines a cohomological Conley index for discrete dynamical systems using the so-called Leray functor. Its construction is based on the notions of a generalized kernel and a generalized image. Now we are going to recall them.

Let $F : A \to A$ denote an endomorphism of a graded module. We define a generalized kernel of F as

$$\operatorname{gker}(F) = \bigcup \{F^{-n}(0)\colon\ n \in \mathbb{N}\}$$

and a generalized image of F as

$$\operatorname{gim}(F) := \bigcap \{F^{n}(A)\colon\ n \in \mathbb{N}\}.$$

Obviously, $\operatorname{gker}(F)$ and $\operatorname{gim}(F)$ are submodules of A.

To define an external multiplication we will use the functor of Alexander–Spanier cohomology H^*, constructed in [8], due to its strong excision property (Lemma 6.4.4). We are also going to apply a cup-product, defined for example in [4] and [2], which will be denoted by $\smile$.

Fix a locally compact metric space X, a continuous map $f : X \to X$, an isolated invariant set S for f, and an index pair $P = (P_1, P_2)$ for S. We consider an index space $(U(P), r, s)$, where $r = r_P$, $s = s_P$. The map f induces

$$f^* : H^*(X) \to H^*(X),$$

while an index map $f_P : (U(P), s(X)) \to (U(P), s(X))$ induces

$$f_P^* : H^*\big(U(P), s(X)\big) \to H^*\big(U(P), s(X)\big)$$

in cohomologies. We denote the set of the fixed points of f^* by $\operatorname{Fix}(f^*)$.

Let $u \in \operatorname{Fix}(f^*) \subseteq H^*(X)$ and $v \in H^*(U(P), s(X))$. By $[v]$ we denote an equivalence class of v in $H^*(U(P), s(X))/\operatorname{gker}(f_P^*)$. We know that $r^*(u) \in H^*(U(P))$ and $r^*(u) \smile v \in H^*(U(P), s(X))$.

We put

$$\widetilde{f_P^*} : H^*\big(U(P), s(X)\big)/\operatorname{gker}(f_P^*) \ni [v] \mapsto \big[f_P^*(v)\big] \in H^*\big(U(P), s(X)\big)/\operatorname{gker}(f_P^*).$$

$\widetilde{f_P^*}$ is a well-defined monomorphism because

$$v \in \operatorname{gker}(f_P^*) \Rightarrow f_P^*(v) \in \operatorname{gker}(f_P^*)$$

and

$$[v] \in \ker(\widetilde{f_P^*}) \Rightarrow f_P^*(v) \in \operatorname{gker}(f_P^*) \Rightarrow v \in \operatorname{gker}(f_P^*) \Rightarrow [v] = \big[\operatorname{gker}(f_P^*)\big].$$

Its restriction

$$\widetilde{f_P^*}|_{\operatorname{gim}(\widetilde{f_P^*})} : \operatorname{gim}(\widetilde{f_P^*}) \to \operatorname{gim}(\widetilde{f_P^*})$$

is a well-defined isomorphism as $\widetilde{f_P^*}(\operatorname{gim}(\widetilde{f_P^*})) \subseteq \operatorname{gim}(\widetilde{f_P^*})$ and $\operatorname{gim}(\widetilde{f_P^*}) \subseteq \operatorname{im}(\widetilde{f_P^*})$.

Now we are ready to define an external multiplication.

Definition 5.1. The external multiplication

$$\operatorname{Fix}(f^*) \times \operatorname{gim}(\widetilde{f_P^*}) \to \operatorname{gim}(\widetilde{f_P^*})$$

is given by:

$$u \cdot [v] := [r^*(u) \smile v].$$

For the proof of the correctness of this definition it is enough to show:

Proposition 5.2.

1. $v \in \operatorname{gker}(f_P^*) \Rightarrow r^*(u) \smile v \in \operatorname{gker}(f_P^*)$.
2. $[v] \in \operatorname{gim}(\widetilde{f_P^*}) \Rightarrow [r^*(u) \smile v] \in \operatorname{gim}(\widetilde{f_P^*})$.

Proof. Take $v \in \operatorname{gker}(f_P^*)$. There exists a $k \in \mathbb{N}$ such that ${f_P^*}^k(v) = 0$. Thus, from the naturalness of the cup-product,

$$ {f_P^*}^k\big(r^*(u) \smile v\big) = {f_P^*}^k\big(r^*(u)\big) \smile {f_P^*}^k(v) = 0, $$

so $r^*(u) \smile v \in \operatorname{gker}(f_P^*)$.

For the proof of point 2 take $[v] \in \operatorname{gim}(\widetilde{f_P^*})$. For any $l \in \mathbb{N}$ there exists a $[w] \in H^*(U(P), s(X))/\operatorname{gker}(f_P^*)$ such that $[v] = {\widetilde{f_P^*}}^l([w]) = [{f_P^*}^l(w)]$. Consequently, $v - {f_P^*}^l(w) \in \operatorname{gker}(f_P^*)$ and from the point 1 we get

$$ r^*(u) \smile \big(v - {f_P^*}^l(w)\big) = \big(r^*(u) \smile v\big) - \big(r^*(u) \smile {f_P^*}^l(w)\big) \in \operatorname{gker}(f_P^*). $$

As $u \in \operatorname{Fix}(f^*)$, we have

$$ \begin{aligned} \big[r^*(u) \smile v\big] &= \big[r^*(u) \smile {f_P^*}^l(w)\big] = \big[r^*\big({f^*}^l(u)\big) \smile {f_P^*}^l(w)\big] \\ &= \big[{f_P^*}^l\big(r^*(u)\big) \smile {f_P^*}^l(w)\big] = \big[{f_P^*}^l\big(r^*(u) \smile w\big)\big] \\ &= {\widetilde{f_P^*}}^l\big(\big[r^*(u) \smile w\big]\big), \end{aligned} $$

which implies that $[r^*(u) \smile v] \in \operatorname{gim}(\widetilde{f_P^*})$. $\square$

Remark 5.3. An object $\operatorname{gim}(\widetilde{f_P^*})$ looks quite unnatural. However, if we study [6], we will see that, in fact, it can be identified with the cohomology Conley index constructed in that paper. So, we have just defined an operation on the index.

Notice that a continuous map

$$ \Psi : \big(U(P'), s'(X)\big) \to \big(U(P), s(X)\big) $$

for which $\Psi^* \circ f_P^* = f'^*_{P'} \circ \Psi^*$, induces a homomorphism

$$ \widetilde{\Psi^*} : \operatorname{gim}(\widetilde{f_P^*}) \ni [v] \mapsto \big[\Psi^*(v)\big] \in \operatorname{gim}(\widetilde{f'^*_{P'}}). $$

The correctness of its definition is guaranteed by the following proposition:

Proposition 5.4.

1. $v \in \operatorname{gker}(f_P^*) \Rightarrow \Psi^*(v) \in \operatorname{gker}(f'^*_{P'})$.
2. $[v] \in \operatorname{gim}(\widetilde{f_P^*}) \Rightarrow [\Psi^*(v)] \in \operatorname{gim}(\widetilde{f'^*_{P'}})$.

Proof. The proof is very similar to the proof of Proposition 5.2, therefore we omit it. $\square$

Calculate

$$ \begin{aligned} \widetilde{f_P^*}\big(u \cdot [v]\big) &= \widetilde{f_P^*}\big(\big[r^*(u) \smile v\big]\big) = \big[f_P^*\big(r^*(u) \smile v\big)\big] = \big[f_P^*\big(r^*(u)\big) \smile f_P^*(v)\big] \\ &= \big[r^*\big(f^*(u)\big) \smile f_P^*(v)\big] = f^*(u) \cdot \big[f_P^*(v)\big] = u \cdot \big[f_P^*(v)\big] \\ &= u \cdot \widetilde{f_P^*}\big([v]\big). \end{aligned} $$

We have just shown

Proposition 5.5. *If $u \in \operatorname{Fix}(f^*)$ and $[v] \in \operatorname{gim}(\widetilde{f_P^*})$, then*

$$ \widetilde{f_P^*}\big(u \cdot [v]\big) = u \cdot \widetilde{f_P^*}\big([v]\big). $$

Theorem 5.6. *The external multiplication $\cdot$ provides $\operatorname{gim}(\widetilde{f_P^*})$ with the structure of a left module over $\operatorname{Fix}(f^*)$, which is invariant under continuation. In particular, it is independent on the choice of an index pair for a fixed isolated invariant set S.*

Proof. For any continuous map $f : X \to X$ we have $1 \in \mathrm{Fix}(f^*)$ so, by the properties of the cup-product, it follows

$$1 \cdot [v] = [v],$$
$$(u_1 \smile u_2) \cdot [v] = u_1 \cdot \big(u_2 \cdot [v]\big),$$

and the external multiplication $\cdot$ provides $\mathrm{gim}(\widetilde{f_P^*})$ with the structure of a left module over $\mathrm{Fix}(f^*)$.

Fix two continuous maps $f, f' : X \to X$, the isolated invariant sets S, S', respectively for f and f', and index pairs P and P' for S and S'. Assume $(f, S) \simeq (f', S')$. Obviously,

$$f^* = f'^* \tag{1}$$

and by the continuation property of the Conley index over a phase space there exist $m, n, k \in \mathbb{N}$ and continuous maps

$$\Phi : \big(U(P), s(X)\big) \to \big(U(P'), s'(X)\big),$$
$$\Psi : \big(U(P'), s'(X)\big) \to \big(U(P), s(X)\big),$$
$$\varphi : X \to X,$$
$$\psi : X \to X,$$

such that

$$\Phi^* \circ f'^*_{P'} = f^*_P \circ \Phi^*, \tag{2}$$
$$\Psi^* \circ f^*_P = f'^*_{P'} \circ \Psi^*, \tag{3}$$
$$\varphi^* = f^{*m}, \tag{4}$$
$$\psi^* = f'^{*n}, \tag{5}$$
$$\Phi^* \circ r'^* = r^* \circ \varphi^*, \tag{6}$$
$$\Psi^* \circ r^* = r'^* \circ \psi^*, \tag{7}$$
$$\Phi^* \circ \Psi^* \circ f_P^{*k} = f_P^{*m+n+k}, \tag{8}$$
$$\Psi^* \circ \Phi^* \circ f'^{*k}_{P'} = f'^{*m+n+k}_{P'}. \tag{9}$$

Conditions (2) and (3) guarantee the existence of maps

$$\widetilde{\Phi^*} : \mathrm{gim}(\widetilde{f'^*_{P'}}) \to \mathrm{gim}(\widetilde{f^*_P}),$$
$$\widetilde{\Psi^*} : \mathrm{gim}(\widetilde{f^*_P}) \to \mathrm{gim}(\widetilde{f'^*_{P'}}).$$

Fix $u \in \mathrm{Fix}(f^*)$ and $[v] \in \mathrm{gim}(\widetilde{f^*_P})$. By (1), (5) and (7) it follows

$$\begin{aligned}\widetilde{\Psi^*}\big(u \cdot [v]\big) &= \widetilde{\Psi^*}\big([r^*(u) \smile v]\big) = \big[\Psi^*\big(r^*(u) \smile v\big)\big] = \big[\Psi^*\big(r^*(u)\big) \smile \Psi^*(v)\big] \\ &= \big[r'^*\big(\psi^*(u)\big) \smile \Psi^*(v)\big] = \psi^*(u) \cdot \big[\Psi^*(v)\big] = f'^{*n}(u) \cdot \big[\Psi^*(v)\big] \\ &= f^{*n}(u) \cdot \big[\Psi^*(v)\big] = u \cdot \big[\Psi^*(v)\big] = u \cdot \widetilde{\Psi^*}\big([v]\big).\end{aligned}$$

It means that $\widetilde{\Psi^*}$ is a homomorphism of modules. Similarly, by (1), (4) and (6) one can show that $\widetilde{\Phi^*}$ is a homomorphism of modules.

Proposition 5.5 implies that $\widetilde{f^*_P}$ is an isomorphism of modules. By (3) and (8) we have

$$\widetilde{\Phi^*} \circ \widetilde{f'^*_{P'}}^{-m} \circ \widetilde{\Psi^*} \circ \widetilde{f^*_P}^{-n} = \widetilde{\Phi^*} \circ \widetilde{\Psi^*} \circ \widetilde{f^*_P}^{k} \circ \widetilde{f^*_P}^{-(m+n+k)} = \mathrm{id}_{\mathrm{gim}(\widetilde{f^*_P})}.$$

Similarly, by (2) and (9) it follows that $\widetilde{\Psi^*} \circ \widetilde{f^*_P}^{-n} \circ \widetilde{\Phi^*} \circ \widetilde{f'^*_{P'}}^{-m} = \mathrm{id}_{\mathrm{gim}(\widetilde{f'^*_{P'}})}$, so $\widetilde{\Phi^*} \circ \widetilde{f'^*_{P'}}^{-m}$ and $\widetilde{\Psi^*} \circ \widetilde{f^*_P}^{-n}$ are mutually inverse isomorphisms of modules. □

Corollary 5.7. *If $f \simeq f'$, $u \in \mathrm{Fix}(f^*) = \mathrm{Fix}(f'^*)$ and $u \cdot \mathrm{gim}(\widetilde{f^*_P})$ is not isomorphic with $u \cdot \mathrm{gim}(\widetilde{f'^*_{P'}})$, then (f, S) and (f', S') are not related by continuation.*

Remark 5.8. Notice that for $f \simeq \mathrm{id}_X$ we have $\mathrm{Fix}(f^*) \cong H^*(X)$ and $\mathrm{gim}(\widetilde{f_P^*}) \cong H^*(U(P), s(X))$ so calculations are much simpler.

In what follows we will use the notions of a directed set, a direct system, a compatible collection of homomorphisms, and the (direct) limit of a direct system, which are defined for example in [5].

In [5] one can find a characterization of the kernel of the components of the limit of a direct system:

Lemma 5.9. *If a family (M_∞, ψ_λ) is the limit of a direct system $\mathcal{M} = (M_\lambda, \psi_{\lambda\mu})$, then $\ker \psi_\lambda = \bigcup_{\mu \geqslant \lambda} \ker \psi_{\lambda\mu}$.*

Next three propositions follow immediately from definitions of the notions mentioned above.

Proposition 5.10. *The set $PI(S)$ of index pairs for S with the relation $\leqslant$ given by*

$$(P_1, P_2) \leqslant (P_1', P_2') \Longleftrightarrow P_1 \supseteq P_1'$$

is a directed set.

Proposition 5.11. *$\mathcal{M} = (H^*(P_1), i^*_{P_1 P_1'})_{P, P' \in PI(S)}$ is a direct system, where $i^*_{P_1 P_1'} : H^*(P_1) \to H^*(P_1')$ is a homomorphism induced by inclusion of the first components of the index pairs $P_1' \subseteq P_1$.*

Proposition 5.12. *$(H^*(S), (i^*_{P_1 S})_{P \in PI(S)})$ is a collection of homomorphisms compatible with*

$$\mathcal{M} = (H^*(P_1), i^*_{P_1 P_1'})_{P, P' \in PI(S)},$$

where $i^*_{P_1 S} : H^*(P_1) \to H^*(S)$ is a homomorphism induced by the inclusion $S \subseteq P_1$.

Due to the stiffness of Alexander–Spanier cohomology (Theorem 8.4 in [5]) we have

Proposition 5.13. *A family $(H^*(S), (i^*_{PS})_{P \in PI(S)})$ is the limit of a direct system $\mathcal{M} = (H^*(P), i^*_{PP'})_{P, P' \in PI(S)}$, where $i^*_{PS} : H^*(P) \to H^*(S)$ is a homomorphism induced by the inclusion $S \subseteq P_1$.*

Now we can formulate

Theorem 5.14. *If $u \in \mathrm{Fix}(f^*)$ and $v \in H^*(U(P), s(X))$, then*

$$u \cdot [v] \neq [0] \Rightarrow u|_S \neq 0. \tag{10}$$

Proof. We prove the theorem in two steps.

From Lemma 5.9 it follows that if $u|_S = 0$, then for any index pair $P \in PI(S)$ we have $u|_{P_1} \in \ker i^*_{P_1 S} = \bigcup_{P \leqslant Q} \ker i^*_{P_1 Q_1}$, so there exists a $Q \in PI(S)$ such that $Q \subseteq P$ and $u|_{P_1} \in \ker i^*_{P_1 Q_1}$, which implies that $u|_{Q_1} = 0$.

To finish the proof it is enough to show

$$u \cdot [v] \neq [0] \Rightarrow u|_{P_1} \neq 0,$$

as the external multiplication does not depend on the choice of an index pair.

Let $u \cdot [v] = [r^*(u) \smile v] \neq [0]$. Then $r^*(u) \smile v \neq 0$. Define a map

$$j : (P_1, P_2) \ni x \longmapsto [x, 1]_P \in \big(U(P), s(X)\big).$$

By the strong excision property j induces an isomorphism in cohomology:

$$j^* : H^*\big(U(P), s(X)\big) \to H^*(P_1, P_2),$$

because $U(P) \setminus s(X) \cong P_1 \setminus P_2$. The composition

$$r \circ j : P_1 \to X$$

is an inclusion, hence

$$j^*\big(r^*(u)\big) = (r \circ j)^*(u) = u|_{P_1}.$$

Thus, by the naturalness of the cup-product,

$$u|_{P_1} \smile j^*(v) = j^*\big(r^*(u)\big) \smile j^*(v) = j^*\big(r^*(u) \smile v\big) \neq 0,$$

and finally

$$u|_{P_1} \neq 0. \qquad \square$$

The following corollaries follow immediately from Theorems 5.6 and 5.14.

Corollary 5.15. *If $u \in \mathrm{Fix}(f^*)$ and $v \in \mathrm{gim}(f_P^*)$, then*

$$u \cdot [v] \neq 0 \Rightarrow u|_{S'} \neq 0$$

for any isolated invariant set S' which is related by continuation with S.

Corollary 5.16. *If $u \in \mathrm{Fix}(f^*)$ and $v \in \mathrm{gim}(f_P^*)$, then*

$$u \cdot [v] \neq 0 \Rightarrow S' \neq \emptyset$$

for any isolated invariant set S' which is related by continuation with S.

We finish with an example illustrating how to show the lack of continuation by means of the external multiplication. We extend some ideas from [9]. To simplify notation, we will write φ^k instead of φ^{*k} to denote the homomorphism induced by a map φ in the kth group of Alexander–Spanier cohomology.

Example 5.17. Consider the space $X = \mathbb{R}^3 \setminus Oz$, where $Oz = \{(0,0,z)\colon z \in \mathbb{R}\}$, and $f, f'\colon X \to X$, which are time-one maps for continuous dynamical systems for which there exist hyperbolic periodic orbits winding respectively once and twice around Oz. The periodic orbits S, S' in both cases are isolated invariant sets, while the thin tubes B and B' including these orbits are isolating blocks for S and S'. Assume for a while that T and T' are small. From the Theorem 2.4 there exists a T_0 such that $P = (B \cup B^- \cdot [0, T_0], B^- \cdot [0, T_0])$ and $P' = (B' \cup B'^- \cdot [0, T_0], B'^- \cdot [0, T_0])$ are index pairs for (S, f) and (S', f'). Obviously, $P_1/P_2 \simeq P_1'/P_2' \simeq S^1 \sqcup \{*\}$ and index maps in Szymczak sense do not differ, so Szymczak indices $h_d(S, f)$ and $h_d(S', f')$ for S and S' are equal (for precise definitions see [12]).

We will calculate cohomologies over $\mathbb{Z}$. We have $f \simeq \mathrm{id}_X \simeq f'$, so $\mathrm{Fix}(f^1) \cong \mathrm{Fix}(f'^1) \cong H^1(X) \cong \mathbb{Z}$ and $f^k \cong f'^k \cong \mathrm{id}_{H^k(X)}$. It follows that dividing by a generalized kernel and taking a generalized image does not change spaces. Assume that a Poincaré map preserves orientation of its unstable manifold. Let $k = 0, 1, 2$ be its dimension. If W is a section of B transversal to a periodic orbit, then it is an isolating block for a Poincaré map. moreover, there exist homeomorphisms

$$h\colon \big(W \times S^1, W^- \times S^1\big) \to (B, B^-),$$
$$h'\colon \big(W \times S^1, W^- \times S^1\big) \to (B', B'^-).$$

Inclusions

$$j\colon (B, B^-) \to \big(U(P), s_P(X)\big),$$
$$j'\colon (B', B'^-) \to \big(U(P'), s_{P'}(X)\big)$$

induce isomorphisms in cohomologies. Thus, the following spaces are isomorphic:

$$\begin{aligned} H^k\big(U(P), s_P(X)\big) &\cong H^k\big(U(P'), s_{P'}(X)\big) \\ &\cong H^k\big(W \times S^1, W^- \times S^1\big) \\ &\cong H^k(B, B^-) \cong H^k(B', B'^-) \cong \mathbb{Z}. \end{aligned}$$

Take u—a generator of $\mathrm{Fix}(f^1) \cong \mathrm{Fix}(f'^1)$ and $[v_1]$, $[v_2]$—generators of $H^k(U(P), s_P(X))$ and $H^k(U(P'), s_{P'}(X))$. By the above isomorphisms, $u \cdot [v_1]$ and $u \cdot [v_2]$ may be identified with $h^1 j^1 r^1_P(u) \smile h^k j^k(v_1)$ and $h'^1 j'^1 {r'_P}^1(u) \smile h'^k j'^k(v_2)$. Let $1_W \in H^0(W)$, $1_{S^1} \in H^0(S^1)$, $s \in H^1(S^1)$, $w \in H^k(W, W^-)$ be generators of cohomology groups. We have

$$
\begin{aligned}
&h^1 j^1 r^1_P(u) \in H^1\bigl(W \times S^1\bigr),\\
&h'^1 j'^1 r^1_{P'}(u) \in H^1\bigl(W' \times S^1\bigr),\\
&h^k j^k(v_1) \in H^k\bigl(W \times S^1, W^- \times S^1\bigr),\\
&h'^k j'^k(v_2) \in H^k\bigl(W \times S^1, W^- \times S^1\bigr).
\end{aligned}
$$

By the multiplicativity of a cup-product (Theorem 8.16 in [2]) we get

$$
\begin{aligned}
h^1 j^1 r^1_P(u) \smile h^k j^k(v_1) &= (1_W \times s) \smile (w \times 1_{S^1})\\
&= \pm\,(1_W \smile w) \times (s \smile 1_{S^1}) = \pm w \times s,\\
h'^1 j'^1 r^1_{P'}(u) \smile h'^k j'^k(v_2) &= (1_W \times 2s) \smile (w \times 1_{S^1})\\
&= \pm\,(1_W \smile w) \times (2s \smile 1_{S^1}) = \pm w \times 2s.
\end{aligned}
$$

As $w \times s \in H^{k+1}(W \times S^1, W^- \times S^1)$, we have $u \cdot [v_1] \neq u \cdot [v_2]$. Corollary 5.7 implies the lack of continuation between (S, f) and (S', f'). Now, if we leave the assumption that T and T' are small, still $(S, f) \not\simeq (S', f')$, while $h_d(S, f) = h_d(S', f')$ by the continuation property of h_d.

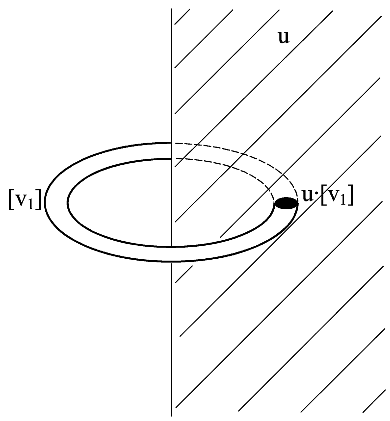

Fig. 1. An attracting orbit ($k = 0$) winding once.

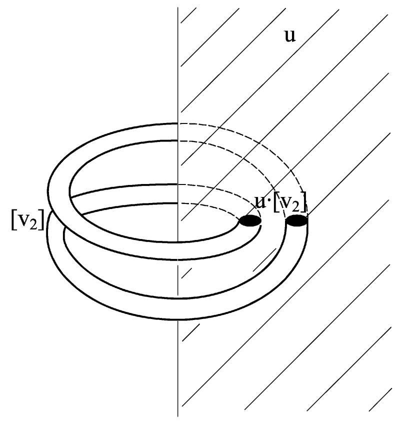

Fig. 2. An attracting orbit ($k = 0$) winding twice.

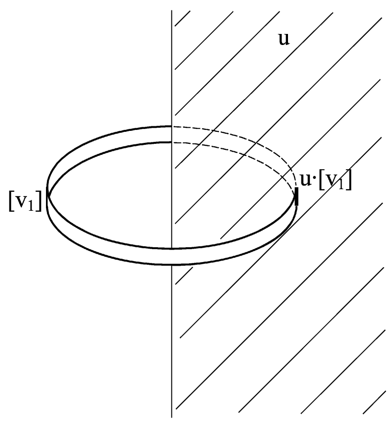

Fig. 3. A saddle orbit ($k = 1$) winding once.

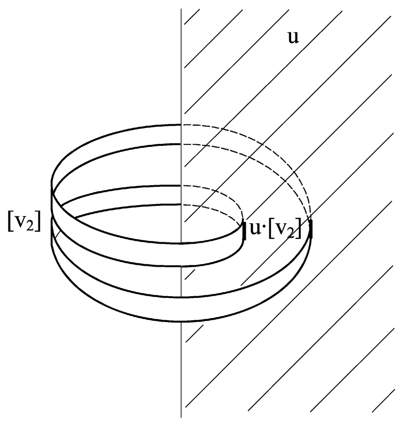

Fig. 4. A saddle orbit ($k = 1$) winding twice.

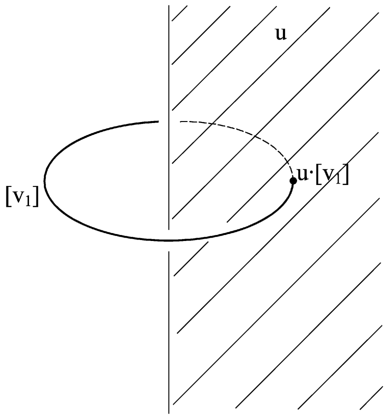

Fig. 5. A repelling orbit ($k = 2$) winding once.

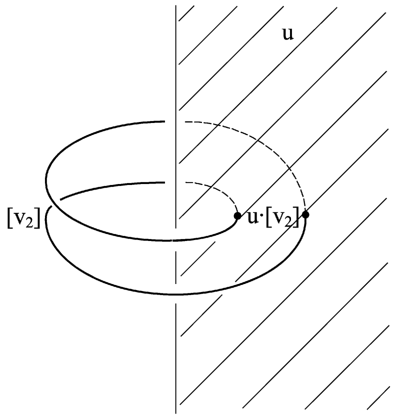

Fig. 6. A repelling orbit ($k = 2$) winding twice.

Geometrically, we can represent cohomology classes as the supports of cocycles.

In the following pictures u is a half-plane, whose edge is the z-axis, while $[v_1]$ and $[v_2]$ are homeomorphic, respectively, with thin tubes around S and S' (Figs. 1 and 2) for $k = 0$, with "tape" including S and S' (Figs. 3 and 4) for $k = 1$, and, finally, with S and S' (Figs. 5 and 6) for $k = 2$.

The supports of cocycles generating products $u \cdot [v_1] \in H^1(U(P), s_P(X))$ and $u \cdot [v_2] \in H^1(U(P'), s_{P'}(X))$ are equal to the intersections of the supports of cocycles u and v_1 as well as of u and v_2. Thus, the first one is: a disc ($k = 0$, Fig. 1), a segment ($k = 1$, Fig. 3), or a point ($k = 2$, Fig. 5). The second one consists of two: discs ($k = 0$, Fig. 2), segments ($k = 1$, Fig. 4), or points ($k = 2$, Fig. 6).

The above example may be easily generalized by considering two hyperbolic periodic orbits in $\mathbb{R}^n$ winding p and q times around one of the coordinate axes.

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